EXAMEN BLANC

Semestre Printemps 2019

Juin 2019

Length of the exam : 3h00 (from12h15 to 15h15)

Attempt all the questions

First write your name, given names and section :

Name :____________________________ **Given Name :** ________________________

Section : ____________________

Exercice	**Points**
1	
2	
3	
4	
5	
Total points :	

1i) True or false (justify) : If for some $i, j \in I$ and positive integer r, $P_{ij}^{r} > 0$ and i is recurrent then j must be also.
(ii) Continuous time irreducible Markov chain on $\{1, 2, \ldots 8\}$, $(X_t)_{t\geq 0}$ has stationary measure λ. Give the stationary distribution for the jump chain in terms of λ and the jump rates $q_1, q_2, \cdots q_8$.
(iii) For discrete time Markov chain $(X_n)_{n\geq 0}$ with $|I| = \{1, 2, \cdots N\}$ for $N < \infty$, we have $\forall\, j \in I$, $\sum_i P_{ij} = 1$. Show that the probability measure $\lambda = (\frac{1}{N}, \frac{1}{N}, \cdots \frac{1}{N})$ is stationary for P. For n very large, what is

$$\frac{1}{n}\sum_{k=1}^{n} cos(X_k)$$

approximately.

(iv) Let $X(t) : t \geq 0$ be poisson process of rate 1 starting at 0. Each time X jumps we roll a (fair) die. What is the distribution of the number of odd numbers rolled in interval $[0, 4]$? Given that on interval $[0, 12]$ there were 13 jumps, what is the probability that exactly 3 6's were rolled by the die ?

(v) For continuous time Markov chain $(X(t))_{t\geq 0}$ on $I = \{1, 2, 3\}$ the jump chain has transition matrix

$$\hat{P}\ \cancel{Q} = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

and the jump rates are $q_i = i$ for $i = 1, 2$ and 3. What is the Q-matrix for X ?

i) True : $P_{jj}^{n} \geq P_{ji}^{s} P_{ii}^{k} P_{ij}^{h}$ with $s+k+h=n$, $P_{ji}^{s} > 0$, $P_{ij}^{h} > 0$

$\Rightarrow \infty = \sum_{k=0}^{\infty} P_{ii}^{k} \leq \frac{1}{P_{ji}^{s} P_{ij}^{h}} \sum_{k} P_{jj}^{s+k+h}$

$\Rightarrow \sum_{l=0}^{\infty} P_{jj}^{l} = \infty$

ii) $$\pi(i) = \frac{\lambda(i)\, q(i)}{\sum_K \lambda(k)\, q_K}$$

iii) $(\lambda P)_j = \sum_i \lambda(i) P_{ij} = \sum_i \frac{1}{N} P_{ij} = \frac{1}{N} \sum_i P_{ij} = \frac{1}{N} = \lambda(j)$.

The chain is supposed to be irreducible:
By the ergodic theorem, we have:

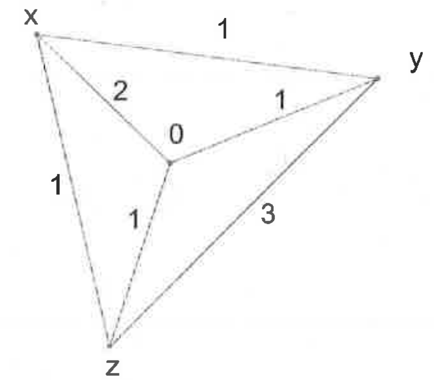
$$\frac{1}{n}\sum_{k=1}^{n}\cos(X_k) \sim \sum_{j=1}^{N}\cos(j)\,\lambda(j) = \frac{1}{N}\sum_{j=1}^{N}\cos(j)$$

iv) $\lambda = 1$, number of jumps in $[0,4] \sim Poi(4)$.
probability of getting an odd number is $\frac{1}{2}$
$\Rightarrow$ number of odd numbers in $[0,4] \sim Poi(2)$.

$$P(3\ 6's) = \binom{13}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{10}.$$

v) $\begin{cases} Q_{ii} = -q_i \\ Q_{ij} = q_i \hat{P}_{ij} \end{cases} \Rightarrow Q: \begin{pmatrix} -1 & \frac{1}{3} & \frac{2}{3} \\ 1 & -2 & 1 \\ 3 & 0 & -3 \end{pmatrix}$

2) For the Markov chain on the graph below the jump rates are as indicated (and only jumps between sites connected by an edge are possible).



(i) Write down the Q matrix. What is the probability that the chain makes three jumps from 0 to x before jumping from 0 to y.

(ii) What is the chance that the chain starting at 0 hits x before y.

(iii) Starting from 0 what is the expected time for the chain to hit x? What is the expected number of times x is visited until the first return to 0 for a chain starting at 0?

(iv) starting from 0 what is the probability that X_t is equal to x for large t (approximately)?

i) $$Q = \begin{array}{c} \\ 0 \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} 0 & x & y & z \end{array} \\ \begin{pmatrix} -4 & 2 & 1 & 1 \\ 2 & -4 & 1 & 1 \\ 1 & 1 & -5 & 3 \\ 1 & 1 & 3 & -5 \end{pmatrix} \end{array}$$

• First interpretation: 3 jumps (at least) from 0 to x before jumping from 0 to y. This probability is $\left(\frac{2}{3}\right)^3$ (where we use the result: $\mathbb{P}(\mathcal{E}xp(\lambda) < \mathcal{E}xp(\mu)) = \frac{\lambda}{\lambda+\mu}$).

• Second interpretation: The first 3 times I'm at o, I jump to x. The fourth time I'm at o, I jump to y.

From o, the probability to jump to x (instead of y or z) is equal to $\frac{2}{4} = \frac{1}{2}$.

Similarly, the probability to jump to y is $\frac{1}{4}$.

$\Rightarrow \mathbb{P}(o \to x, o \to x, o \to x, o \to y) = \left(\frac{1}{2}\right)^3 \times \frac{1}{4}$.

ii) $h(i) := \mathbb{P}(T_x < T_y \mid X_0 = i)$, $i = o, x, y, z$.

$h(x) = 1$, $h(y) = 0$.

$$h(o) = \frac{1}{2} h(x) + \frac{1}{4} h(y) + \frac{1}{4} h(z)$$
$$= \frac{1}{2} + \frac{1}{4} h(z) = \frac{1}{2} + \frac{1}{4}\left(\frac{1}{5} h(x) + \frac{1}{5} h(o) + \frac{3}{5} h(y)\right)$$
$$= \frac{1}{2} + \frac{1}{4}\left(\frac{1}{5} + \frac{1}{5} h(o)\right)$$

$\Rightarrow h(o) = \frac{11}{19}$.

iii) $$K(o) := \mathbb{E}_o(T_x) = \frac{1}{-Q(o,o)} + \sum_{\substack{j \neq o \\ (j = x, y, z)}} \frac{Q(o,j)}{-Q(o,o)} \cdot K(j)$$

$$\Rightarrow K(o) = \frac{1}{4} + \frac{1}{2} \times 0 + \frac{1}{4} K(y) + \frac{1}{4} K(z) \underset{\substack{\text{by symmetry,} \\ K(y) = K(z)}}{=} \frac{1}{4} + \frac{1}{2} K(y).$$

$$K(y) = \frac{1}{5} + \frac{3}{5} K(z) + \frac{1}{5} K(o) + \frac{1}{5} \times 0$$

$$\Longrightarrow 2K(y) = 1 + K(o) \Longrightarrow K(o) = \dots = \frac{2}{3}.$$

$\mathbb{E}[\#\ x \text{ hit before returning to } 0] = \frac{\pi(x)}{\pi(0)}$

(Another method, see Solution - Exercise 3 (Examen Blanc) pdf on moodle in week April 1 → April 7).

So we need to compute π (stationary dist. of the jump chain)

First we find λ such that $\lambda Q = 0$ and $\sum_i \lambda(i) = 1$.

$$\begin{cases} -4\lambda(0) + 2\lambda(x) + \lambda(y) + \lambda(z) = 0 \\ 2\lambda(0) - 4\lambda(x) + \lambda(y) + \lambda(z) = 0 \\ \lambda(0) + \lambda(x) - 5\lambda(y) + 3\lambda(z) = 0 \\ \lambda(0) + \lambda(x) + 3\lambda(y) - 5\lambda(z) = 0 \\ \lambda(y) = \lambda(z) \quad (\text{by symmetry}) \end{cases}$$

$$\Rightarrow \begin{cases} -2\lambda(0) + \lambda(x) + \lambda(y) = 0 \\ \lambda(0) - 2\lambda(x) + \lambda(y) = 0 \\ \lambda(0) + \lambda(x) - 2\lambda(y) = 0 \end{cases}$$

$\Rightarrow \lambda(0) = \lambda(x) = \lambda(y) = \lambda(z)$. $\Rightarrow \lambda = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

By (ii) of exercise 1, $\pi(i) = \dfrac{\lambda(i)\, q(i)}{\sum_k \lambda(k)\, q_k}$

$\Rightarrow$ In this case, $\pi(0) = \dfrac{\lambda(0) \times 4}{\lambda(0) \sum_k q_k} = \dfrac{4}{4+4+5+5} = \frac{4}{18}$.

$\Rightarrow \pi = \left(\frac{4}{18}, \frac{4}{18}, \frac{5}{18}, \frac{5}{18}\right)$

$\Rightarrow \mathbb{E}[\#\ x \text{ hit before returning to } 0] = \frac{\pi(x)}{\pi(0)} = 1$.

(iv) $\mathbb{P}(X_t = k) \xrightarrow{t \to \infty} \lambda(k) = \frac{1}{4}$.

3)

(a) A king moves on a chess board : at each turn he chooses an adjacent square randomly.

(i) How many time steps do we expect to wait until the king reaches its original position ?

(ii) If he starts at the top left corner (denoted i), what is the expected occupation time at the center of the chessboard (the center being the 4 central squares) before the king gets back to its initial position ?

(iii) If j denotes one of the central squares, does one have $\mathbb{E}_j[T_i] = \mathbb{E}_i[T_j]$?

(b) Consider the numbers $1, 2, \cdots, 12$ written around a ring as they usually are on a clock. Consider a Markov chain that at any point jumps with equal probability to the two adjacent numbers.

(i) What is the expected number of steps that X_n will take to return to its starting position ?

(ii) What is the probability that X_n will visit all the other states before returning to its starting position ?

(iii) Starting from 1, what is the expected number of steps needed before reaching 12 ?

a) i) If the starting position is square i, the waiting time before the first return to i is $\frac{1}{\pi(i)}$ on average.

As seen before, $\pi(i) = \frac{d_i}{\sum_k d_k}$ where d_k is the number of legal moves from square k.

3 types of squares: - corner : 3 legal moves $\Rightarrow \pi(\text{corner}) = \frac{3}{\sum_k d_k}$

- border : 5 " "

- remaining squares : 8 " "

$\Rightarrow \sum_k d_k = 3\times 4 + 5 \times 24 + \overset{8}{8} \times 36 = 420$

$\Rightarrow \pi(\text{corner}) = \frac{3}{420}$, $\pi(\text{border}) = \frac{5}{420}$, $\pi(\text{other squares}) = \frac{8}{420}$.

ii) $\dfrac{4 \times \pi(\text{center})}{\pi(\text{top left corner})} = \dfrac{4 \times \frac{8}{420}}{\frac{3}{420}} = \dfrac{4 \times 8}{3} = \dfrac{32}{3}.$

iii) NO

f) i) The chain is doubly stochastic: $P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \cdots & 0 \\ & & \ddots & & \ddots & \end{pmatrix}$

$\Rightarrow$ By exercise 1(iii), the stationary distribution is uniform in this case $\Rightarrow \pi = \left(\frac{1}{12}, \dots, \frac{1}{12}\right)$.

$\Rightarrow \mathbb{E}_i(T_i) = \dfrac{1}{\pi(i)} = 12.$

ii) Without loss of generality, suppose $X_0 = 12$.
In order to visit all the states before returning to 12, we have two possibilities:

- $X_1 = 1$ and we visit 11 before 12 $(T_{11} < T_{12})$
- $X_1 = 11$ and we visit 1 before 12 $(T_1 < T_{12})$

So we have $p = \mathbb{P}_{12}(X_1 = 1) \cdot \mathbb{P}_1(T_{11} < T_{12}) + \mathbb{P}_{12}(X_1 = 11) \times \mathbb{P}_{11}(T_1 < T_{12})$

$= \frac{1}{2} \mathbb{P}_1(T_{11} < T_{12}) + \frac{1}{2} \mathbb{P}_{11}(T_1 < T_{12})$

$\underset{\text{by symmetry}}{=} \mathbb{P}_1(T_{11} < T_{12}).$

Let $h_i =: \mathbb{P}_i(T_{11} < T_{12})$. So $h_{12} = 0$ and $h_{11} = 1$.

Moreover: $h_1 = \frac{1}{2} h_2 + \frac{1}{2} h_{12} = \frac{1}{2} h_2 = \frac{1}{2}\left(\frac{1}{2} h_1 + \frac{1}{2} h_3\right)$

$= \frac{1}{4} h_1 + \frac{1}{4} h_3$

$\Rightarrow h_1 = \frac{1}{3} h_3$.

By similar computations, we get $\cancel{h_1} = \frac{1}{i} \cancel{h_i}$, $i \neq 11, 12$.

$\Rightarrow \cancel{g}\, h_1 = \frac{1}{10} h_{10} = \frac{1}{10}\left(\frac{1}{2} h_{11} + \frac{1}{2} h_9\right)$

$= \frac{1}{10}\left(\frac{1}{2} + \frac{1}{2} \times 9 h_1\right) = \frac{1}{20} + \frac{9}{20} h_1$

$(h_{11} = 1)$

$\Rightarrow h_1 = \frac{1}{11}$

$\Rightarrow \boxed{P = h_1 = \frac{1}{11}}$.

iii) $g(i) := \mathbb{E}_i[T_{12}] \Rightarrow g_{12} = 0$.

$g(1) = 1 + \frac{1}{2} g(2)$.

$g(2) = 1 + \frac{1}{2} g(1) + \frac{1}{2} g(3)$

$g(3) = 1 + \frac{1}{2} g(2) + \frac{1}{2} g(4)$

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We get $g(1) = 11$.

5)

(a) A computer lab has three laser printers that are hooked to the network. A working printer will function for an exponential amount of time with mean 20 days. Upon failure it is immediately sent to the repair facility. There machines are worked on by two repairman who can each repair one printer in an exponential amount of time with mean 2 days. However, it is not possible for two people to work on one printer at once.

   (i) Formulate a Markov chain model for the number of working printers and find the stationary distribution.

   (ii) How often are both repairmen busy?

   (iii) What is the average number of machines in use?

(b) A small company maintains a fleet of four cars to be driven by its workers on business trips. Requests to use cars are a Poisson process with rate 1.5 per day. A car is used for an exponentially distributed time with mean 2 days. Forgetting about weekends, we arrive at the following Markov chain for the number of cars in service :

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} -1.5 & 1.5 & 0 & 0 & 0 \\ 0.5 & -2 & 1.5 & 0 & 0 \\ 0 & 1 & -2.5 & 1.5 & 0 \\ 0 & 0 & 1.5 & -3 & 1.5 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix} \end{array}$$

   (i) Find the stationary distribution.

   (ii) At what rate do unfulfilled requests come in? How would this change if there were only three cars?

   (iii) Let $g(i) = \mathbb{E}_i[T_4]$. Write and solve equations to find the $g(i)$.

   (iv) Use the stationary distribution to compute $\mathbb{E}_3[T_4]$.

a) i) $I = \{0, 1, 2, 3\}$, X_t = # of working machines at time t.

$\lambda = \frac{1}{20}$, $\mu = \frac{1}{2}$.

$$Q = \begin{pmatrix} -2\mu & 2\mu & 0 & 0 \\ \lambda & -\lambda-2\mu & 2\mu & 0 \\ 0 & 2\lambda & -2\lambda-\mu & \mu \\ 0 & 0 & 3\lambda & -3\lambda \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 & 0 \\ \frac{1}{20} & -\frac{1}{20}-1 & 1 & 0 \\ 0 & \frac{1}{10} & -\frac{1}{10}-\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{3}{20} & -\frac{3}{20} \end{pmatrix}$$

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We can use the detailed balance equations to find the stationary distribution in this case:

$$\begin{cases} \pi(0)\, Q(0,1) = \pi(1)\, Q(1,0) \\ \pi(1)\, Q(1,2) = \pi(2)\, Q(2,1) \\ \pi(2)\, Q(2,3) = \pi(3)\, Q(3,2) \end{cases} \quad \text{and} \quad \sum_i \pi(i) = 1$$

We get $\pi(1) = 20\, \pi(0)$, $\pi(2) = 200\, \pi(0)$,

$$\pi(3) = \frac{2000}{3}\, \pi(0)$$

$$\sum_i \pi(i) = 1 \Rightarrow \pi = \frac{3}{2663}\left(1, 20, 200, \frac{2000}{3}\right)$$

ii) Both busy when we are at state 0 or 1.

$\Rightarrow$ Proportion of time $= \pi(0) + \pi(1) = \dfrac{63}{2663}$.

iii) $M :=$ Average number of machines in use.

$$M = 0 \cdot \pi(0) + 1 \cdot \pi(1) + 2\, \pi(2) + 3\, \pi(3) = \frac{7260}{2663}.$$

b) i) $\pi Q = 0 \Rightarrow$

$$\begin{cases} \pi(0) = \frac{1}{3}\pi(1) \\ 2\pi(1) = \frac{1}{2}\pi(1) + \pi(2) \Rightarrow \pi(2) = \frac{3}{2}\pi(1). \\ 5\pi(2) = 3\pi(1) + 3\pi(3) \Rightarrow \pi(3) = \frac{3}{2}\pi(1). \\ 3\pi(3) = 2\pi(4) + \frac{3}{2}\pi(2) \Rightarrow \pi(4) = \frac{9}{8}\pi(1). \end{cases}$$

$$\sum_i \pi(i) = 1 \Rightarrow \pi(1)\left(\frac{1}{3} + 1 + \frac{3}{2} + \frac{3}{2} + \frac{9}{8}\right) = 1 \Rightarrow \pi(1) = \frac{24}{131}$$

$$\pi = \left(\frac{8}{131}, \frac{24}{131}, \frac{36}{131}, \frac{36}{131}, \frac{27}{131}\right).$$

ii) Proportion of time in state 4 is $\pi(4)$ (in the long run).

Requests to use cars follow a Poisson process with rate $\frac{3}{2}$.

$\Rightarrow$ Rate of unfulfilled requests: $\frac{3}{2} \cdot \pi(4)$

If there were only 3 cars, this rate would increase.

iii) $g(0) = \frac{1}{1{,}5} + g(1) = \frac{2}{3} + g(1)$

$g(1) = \frac{1}{2} + \frac{0{,}5}{2} g(0) + \frac{1{,}5}{2} g(2) = \frac{1}{2} + \frac{1}{4} g(0) + \frac{3}{4} g(2)$

$g(2) = \frac{2}{5} + \frac{2}{5} g(1) + \frac{3}{5} g(3)$.

$g(3) = \frac{1}{3} + \frac{1}{2} g(2) + \underbrace{\frac{1}{2} g(4)}_{0}$ ($g(4) = 0$ in this case)

$\Rightarrow g(2) = \frac{2}{5} + \frac{2}{5} g(1) + \frac{1}{5} + \frac{3}{10} g(2)$

$\Rightarrow \frac{7}{10} g(2) = \frac{3}{5} + \frac{2}{5} g(1) \Rightarrow g(2) = \frac{6}{7} + \frac{4}{7} g(1)$.

$g(1) = \frac{1}{2} + \frac{1}{4}\left(\frac{2}{3} + g(1)\right) + \frac{3}{4}\left(\frac{6}{7} + \frac{4}{7} g(1)\right) = \frac{1}{2} + \frac{1}{6} + \frac{1}{4} g(1) + \frac{9}{14} + \frac{3}{7} g(1)$

$\Rightarrow g(1) = \frac{110}{27}$

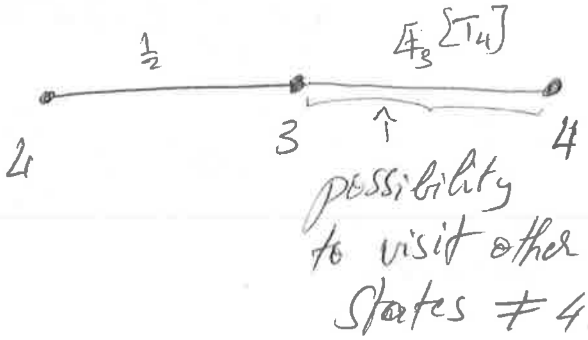
$g(2) = \frac{86}{27}$

$g(3) = \frac{52}{27}$

$g(0) = \frac{2}{3} + \frac{110}{27} = \frac{128}{27}$

iv) Starting at 4, we need on average $\frac{1}{2}$ amount of time ($\mathbb{E}[\mathrm{Exp}(2)] = \frac{1}{2}$) to leave 4 and go to 3 (only possibility in this case).

Once in 3, we need on average $\mathbb{E}_3[T_4]$ time before returning to 4.



So the proportion of time spent at 4 is

$$\frac{\frac{1}{2}}{\frac{1}{2} + \mathbb{E}_3[T_4]}.$$

But this proportion should be equal to $\pi(4)$.

So we have: $\pi(4) = \dfrac{\frac{1}{2}}{\frac{1}{2} + \mathbb{E}_3[T_4]} = \dfrac{1}{1 + 2\mathbb{E}_3[T_4]}$

$$\to \mathbb{E}_3[T_4] = \frac{1}{2}\left(\frac{1}{\pi(4)} - 1\right) = \frac{1}{2}\left(\frac{131}{27} - 1\right)$$

$$= \frac{52}{27} \checkmark$$